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Some new properties of Jacobi's theta functions

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Abstract

In this paper, a monotonicity property for the quotient of two Jacobi's theta functions with respect to the modulus k is proved.

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1. Introduction and main result

Let $k \in (0, 1)$ denote the modulus of Jacobi's elliptic functions $\mathrm{sn}(u)$, $\mathrm{cn}(u)$, and $\mathrm{dn}(u)$, of Jacobi's theta functions $\Theta(u)$, $H(u)$, $H_1(u)$, and $\Theta_1(u)$, and, finally, of Jacobi's zeta function, $\mathrm{zn}(u)$. Here we follow the notation of Carlson and Todd [3], in other references, like [8], Jacobi's zeta function is denoted by $Z(u)$. Let $k' := \sqrt{1-k^2} \in (0, 1)$ be the complementary modulus, let $K \equiv K(k)$ and $E \equiv E(k)$ be the complete elliptic integral of the first and second kind, respectively, and let $K' \equiv K'(k) := K(k')$ and $E' \equiv E'(k) := E(k')$.

If we want to point out the dependence of these functions on the modulus k, we will write $K(k)$, $K'(k)$, $\mathrm{sn}(u, k)$, $\Theta(u, k)$, etc. In this paper, we follow the old notation of the theta functions, which goes back to Jacobi. There is another notation of the four theta functions given by $\Theta(u, k) = \vartheta_0(v, \tau) = \vartheta_4(v, \tau)$, $H(u, k) = \vartheta_1(v, \tau)$, $H_1(u, k) = \vartheta_2(v, \tau)$ and $\Theta_1(u, k) = \vartheta_3(v, \tau)$, where $\tau = \mathrm{i}K'/K$ and $v = u/(2K)$ (note that in some references, like [2] and [4], $v = u\pi/(2K)$).

For the definitions and many important properties of these functions, see, e.g., [2,4,5,7,1].

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In [3], Carlson and Todd proved that, for each $\lambda \in (0,1)$, the functions $\mathrm{sn}(\lambda K, k)$ and $\mathrm{zn}(\lambda K, k)$ are strictly monotone increasing with respect to the modulus k, $0<k<1$. In addition, they investigated the degenerating behaviour of these functions as $k \to 0$ and especially as $k \to 1$. Hence the question arises, whether analogous monotonicity properties hold for the theta functions. Unfortunately, $\Theta(\lambda K, k)$ does not have the same monotonicity behaviour with respect to the modulus k for every $\lambda \in (0,1)$. Numerical examples show that for small λ ($\lambda \leqslant 0.5$), $\Theta(\lambda K)$ is strictly monotone decreasing, for large λ ($\lambda \geqslant 0.6$), $\Theta(\lambda K)$ is strictly monotone increasing in k and for some $\lambda \in (0.5, 0.6)$, $\Theta(\lambda K)$ is not monotone at all in the whole interval $(0,1)$. However, we are able to prove that for each $\lambda, \mu \in \mathbb{R}$, the quotient $\Theta(\lambda K)/\Theta(\mu K)$ of two theta functions is strictly monotone. In addition, the degenerative behaviour of $\Theta(\lambda K)/\Theta(\mu K)$ as $k \to 0$ and $k \to 1$ is given.

Theorem 1. (i) *Let* $\lambda, \mu \in \mathbb{R}$. *If* $\cos(\lambda\pi) > \cos(\mu\pi)$ $[\cos(\lambda\pi) < \cos(\mu\pi)]$, *then* $\Theta(\lambda K)/\Theta(\mu K)$ *is a positive, strictly monotone decreasing* [*increasing*] *function of the modulus* $k \in (0,1)$. *If* $\cos(\lambda\pi) = \cos(\mu\pi)$, *i.e.* $\lambda = \mu + 2\nu$, $\nu \in \mathbb{Z}$, *then* $\Theta(\lambda K)/\Theta(\mu K) = 1$.

(ii) *Let* $\lambda, \mu \in (0,1)$, $k \in (0,1)$, *then* $\Theta(\lambda K)/\Theta(\mu K) \to 1$ *as* $k \to 0$ *and*

$$\frac{\Theta(\lambda K)}{\Theta(\mu K)} \sim \left(\frac{k'}{4}\right)^{(\mu-\lambda)(1-(\lambda+\mu)/2)} \qquad \text{as } k \to 1.$$

(iii) *Let* $\mu \in (0,1)$ *and* $k \in (0,1)$. *Then* $f(\lambda) := \Theta((\mu-\lambda)K)/\Theta((\mu+\lambda)K)$ *is a convex function of* $\lambda \in (0,1)$ *with* $f(0) = f(1) = 1$.

Remark. By the relation $\Theta_1(u) = \Theta(u+K)$, one gets analogous monotonicity properties for the quotients $\Theta_1(\lambda K)/\Theta_1(\mu K)$, $\Theta(\lambda K)/\Theta_1(\mu K)$, and $\Theta_1(\lambda K)/\Theta(\mu K)$.

In [6], we considered polynomials, whose $[-1,1]$ inverse image consists of two Jordan arcs, i.e., we characterized polynomials P_n, for which $P_n^{-1}([-1,1])$ consists of two Jordan arcs (in general, $P_n^{-1}([-1,1])$ consists of n Jordan arcs). Since these polynomials P_n can be given with the help of an elliptic integral, Jacobi's elliptic and theta functions appear in a natural way. When describing the shape of the two Jordan arcs, we need the above theorem, see [6, Theorem 22, Lemma 33 and Lemma 34].

2. Proof of the main result

First, we collect some derivation formulas, which are an immediate consequence of formula (710.00) of [2].

Lemma 2. *Let* $k \in (0,1)$. *Then*

$$\frac{\mathrm{d}K}{\mathrm{d}k} = \frac{E - k'^2 K}{kk'^2}, \qquad \frac{\mathrm{d}K'}{\mathrm{d}k} = \frac{k^2 K' - E'}{kk'^2},$$

$$\frac{\mathrm{d}}{\mathrm{d}k}\left\{\frac{1}{K}\right\} = \frac{k'^2 K - E}{kk'^2 K^2}, \qquad \frac{\mathrm{d}}{\mathrm{d}k}\left\{\frac{K'}{K}\right\} = \frac{-\pi}{2kk'^2 K^2}.$$

Concerning the limiting behaviour of $\mathrm{sn}(\lambda K)$, etc., as $k \to 1$, Carlson and Todd [3] have proved the following.

Lemma 3. *Let* $0 < \lambda < 1$, *then, as* $k \to 1$, *we have*

$$K \sim \log\left(\frac{4}{k'}\right), \quad \mathrm{sn}(\lambda K) \sim 1 - 2\left(\frac{k'}{4}\right)^{2\lambda}, \quad \mathrm{cn}(\lambda K) \sim \mathrm{dn}(\lambda K) \sim 2\left(\frac{k'}{4}\right)^{\lambda},$$
$$\mathrm{zn}(\lambda K) \sim 1 - \lambda - 2\left(\frac{k'}{4}\right)^{2\lambda}.$$

Moreover, we will need the following formulas for the derivatives of the theta functions, which are a direct consequence of (1053.01), (1052.02) and (731.01)–(731.03) of [2].

Lemma 4. *The following relations hold*:

$$\frac{\partial}{\partial u}\{\Theta(u)\} = \Theta(u)\mathrm{zn}(u),$$
$$\frac{\partial}{\partial u}\{H(u)\} = \sqrt{k}\,\Theta(u)(\mathrm{cn}(u)\mathrm{dn}(u) + \mathrm{sn}(u)\mathrm{zn}(u)),$$
$$\frac{\partial}{\partial u}\{H_1(u)\} = \frac{\sqrt{k}}{\sqrt{k'}}\,\Theta(u)(-\mathrm{sn}(u)\mathrm{dn}(u) + \mathrm{cn}(u)\mathrm{zn}(u)),$$
$$\frac{\partial}{\partial u}\{\Theta_1(u)\} = \frac{1}{\sqrt{k'}}\,\Theta(u)(-k^2\mathrm{sn}(u)\mathrm{cn}(u) + \mathrm{dn}(u)\mathrm{zn}(u))$$

and

$$\frac{\partial^2}{\partial u^2}\{\Theta(u)\} = \Theta(u)(\mathrm{dn}^2(u) + \mathrm{zn}^2(u) - E/K),$$
$$\frac{\partial^2}{\partial u^2}\{H(u)\} = \sqrt{k}\,\Theta(u)(-k^2\mathrm{sn}(u)\mathrm{cn}^2(u) + 2\mathrm{cn}(u)\mathrm{dn}(u)\mathrm{zn}(u) + \mathrm{sn}(u)(\mathrm{zn}^2(u) - E/K)),$$
$$\frac{\partial^2}{\partial u^2}\{H_1(u)\} = \frac{\sqrt{k}}{\sqrt{k'}}\,\Theta(u)(-k^2\mathrm{sn}^2(u)\mathrm{cn}(u) - 2\mathrm{sn}(u)\mathrm{dn}(u)\mathrm{zn}(u) + \mathrm{cn}(u)(\mathrm{zn}^2(u) - E/K)),$$
$$\frac{\partial^2}{\partial u^2}\{\Theta_1(u)\} = \frac{1}{\sqrt{k'}}\,\Theta(u)(\mathrm{dn}(u)(1-k^2\mathrm{cn}^2(u))-2k^2\mathrm{sn}(u)\mathrm{cn}(u)\mathrm{zn}(u)+\mathrm{dn}(u)(\mathrm{zn}^2(u)-E/K)).$$

Remark. By (123.01), (123.03) of [2] and Lemma 4, the relation

$$\frac{\partial}{\partial u}\left\{\log\left(\frac{H(v-u)}{H(v+u)}\right)\right\} = \frac{2\,\mathrm{sn}(v)\,\mathrm{cn}(v)\,\mathrm{dn}(v)}{\mathrm{sn}^2(u) - \mathrm{sn}^2(v)} - 2\,\mathrm{zn}(v)$$

holds which implies

$$\log\left(\frac{H(v-u)}{H(v+u)}\right) = \int_0^u \frac{2\,\mathrm{sn}(v)\,\mathrm{cn}(v)\,\mathrm{dn}(v)}{\mathrm{sn}^2(u) - \mathrm{sn}^2(v)}\,\mathrm{d}u - 2\,u\,\mathrm{zn}(v). \tag{1}$$

Analogous formulas can be obtained for other ratios of theta functions.

The next lemma gives the derivatives with respect to the modulus k for the four theta functions $\Theta(u)$, $H(u)$, $H_1(u)$, $\Theta_1(u)$, where $u=\lambda K$. Note that λK as well as the theta functions themselves depend on the modulus k.

Lemma 5. *For $\lambda\in\mathbb{R}$, the derivatives with respect to k of the four theta functions $\Theta(\lambda K)$, $H(\lambda K)$, $H_1(\lambda K)$, $\Theta_1(\lambda K)$ are given by*

$$\frac{\mathrm{d}}{\mathrm{d}k}\{\Theta(\lambda K)\}=-\frac{1}{2kk'^2}\frac{\partial^2}{\partial u^2}\{\Theta(u)\}\bigg|_{u=\lambda K}=-\frac{1}{2kk'^2}\,\Theta(\lambda K)(\mathrm{dn}^2(\lambda K)+\mathrm{zn}^2(\lambda K)-E/K),$$

$$\frac{\mathrm{d}}{\mathrm{d}k}\{H(\lambda K)\}=-\frac{1}{2kk'^2}\frac{\partial^2}{\partial u^2}\{H(u)\}\bigg|_{u=\lambda K},$$

$$\frac{\mathrm{d}}{\mathrm{d}k}\{H_1(\lambda K)\}=-\frac{1}{2kk'^2}\frac{\partial^2}{\partial u^2}\{H_1(u)\}\bigg|_{u=\lambda K},$$

$$\frac{\mathrm{d}}{\mathrm{d}k}\{\Theta_1(\lambda K)\}=-\frac{1}{2kk'^2}\frac{\partial^2}{\partial u^2}\{\Theta_1(u)\}\bigg|_{u=\lambda K}.$$

Proof. The four theta functions $\vartheta_j(v,\tau)$, $j=1,2,3,4$, satisfy a differential equation of the form

$$\frac{\partial^2}{\partial v^2}\{\vartheta_j(v,\tau)\}=4\mathrm{i}\pi\frac{\partial}{\partial\tau}\{\vartheta_j(v,\tau)\},$$

see [5, p. 375]. By Lemma 2 and since $\Theta(u,k)=\vartheta_4(\frac{u}{2K},\tau)$, where $\tau=\mathrm{i}K'/K$,

$$\begin{aligned}\frac{\partial}{\partial k}\{\Theta(u,k)\}&=\frac{\partial}{\partial v}\{\vartheta_4(\tfrac{u}{2K},\tau)\}\frac{\mathrm{d}}{\mathrm{d}k}\left\{\frac{u}{2K}\right\}+\frac{\partial}{\partial\tau}\{\vartheta_4(\tfrac{u}{2K},\tau)\}\frac{\mathrm{d}\tau}{\mathrm{d}k}\\&=\frac{u(k'^2K-E)}{kk'^2K}\frac{\partial}{\partial u}\{\Theta(u,k)\}-\frac{1}{2kk'^2}\frac{\partial^2}{\partial u^2}\{\Theta(u,k)\}.\end{aligned}$$

Thus, for the derivative with respect to k, by Lemma 2, we get

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}k}\{\Theta(\lambda K,k)\}&=\frac{\mathrm{d}}{\mathrm{d}k}\{\lambda K\}\frac{\partial}{\partial u}\{\Theta(u,k)\}\bigg|_{u=\lambda K}+\frac{\partial}{\partial k}\{\Theta(u,k)\}\bigg|_{u=\lambda K}\\&=\frac{\lambda(E-k'^2K)}{kk'^2}\frac{\partial}{\partial u}\{\Theta(u,k)\}\bigg|_{u=\lambda K}+\frac{\lambda K(k'^2K-E)}{kk'^2K}\frac{\partial}{\partial u}\{\Theta(u,k)\}\bigg|_{u=\lambda K}\\&\quad-\frac{1}{2kk'^2}\frac{\partial^2}{\partial u^2}\{\Theta(u,k)\}\bigg|_{u=\lambda K}=-\frac{1}{2kk'^2}\frac{\partial^2}{\partial u^2}\{\Theta(u,k)\}\bigg|_{u=\lambda K}.\end{aligned}$$

Since the last identity holds for the other three theta functions as well, this gives the assertion. □

Proof of Theorem 1. (i) By Lemma 5,

$$\frac{\mathrm{d}}{\mathrm{d}k}\left\{\frac{\Theta(\lambda K)}{\Theta(\mu K)}\right\}=-\frac{1}{2kk'^2}\frac{\Theta(\lambda K)}{\Theta(\mu K)}(\mathrm{dn}^2(\lambda K)+\mathrm{zn}^2(\lambda K)-\mathrm{dn}^2(\mu K)-\mathrm{zn}^2(\mu K)).$$

Thus, it remains to be shown that $g(u) := \mathrm{dn}^2(u)+\mathrm{zn}^2(u)$ satisfies the inequality $g(\lambda K)-g(\mu K) > 0\,[<0]$ for all $\lambda, \mu \in \mathbb{R}$ with $\cos(\lambda\pi) > [<] \cos(\mu\pi)$. This property holds since $g(-u) = g(u)$, $g(u+2K) = g(u)$ and $g(u)$ is a positive, strictly monotone decreasing function in $(0, K)$. Concerning the monotonicity of $g(u)$, note that

$$g'(u) = -2\mathrm{dn}^2(u)\underbrace{\left(\frac{k^2\mathrm{sn}(u)\mathrm{cn}(u)}{\mathrm{dn}(u)} - \mathrm{zn}(u)\right)}_{=:h(u)} - \frac{2\,E\,\mathrm{zn}(u)}{K},$$

where $h(0) = h(K) = 0$ and

$$h''(u) = -\frac{2k^2k'^2\,\mathrm{sn}(u)\mathrm{cn}(u)}{\mathrm{dn}^3(u)} < 0 \quad \text{for } u \in (0, K).$$

Thus $h(u) > 0$ and therefore $g'(u) < 0$ for $u \in (0, K)$.

(ii) Assume that $\lambda < \mu$ and let $\alpha := (\lambda+\mu)/2$, $\beta := (\mu-\lambda)/2$, i.e. $\lambda = \alpha - \beta$, $\mu = \alpha + \beta$, $\alpha, \beta \in (0, 1)$. By (1052.02) of [2] and (1), we get

$$\begin{aligned}\log\left(\frac{\Theta(\lambda K)}{\Theta(\mu K)}\right) &= \log\left(\frac{\mathrm{sn}(\mu K)H(\lambda K)}{\mathrm{sn}(\lambda K)H(\mu K)}\right) = \log(\mathrm{sn}(\mu K)) - \log(\mathrm{sn}(\lambda K))\\ &\quad + 2K\,\mathrm{sn}(\alpha K)\mathrm{cn}(\alpha K)\mathrm{dn}(\alpha K)\int_0^\beta \frac{\mathrm{d}v}{\mathrm{sn}^2(vK) - \mathrm{sn}^2(\alpha K)} - 2\beta K\,\mathrm{zn}(\alpha K).\end{aligned}$$

Thus, as $k \to 0$,

$$\begin{aligned}\log\left(\frac{\Theta(\lambda K)}{\Theta(\mu K)}\right) &\sim \log\left(\sin\left(\frac{\mu\pi}{2}\right)\right) - \log\left(\sin\left(\frac{\lambda\pi}{2}\right)\right)\\ &\quad + \pi\sin\left(\frac{\alpha\pi}{2}\right)\cos\left(\frac{\alpha\pi}{2}\right)\int_0^\beta \frac{\mathrm{d}v}{\sin^2(\frac{v\pi}{2}) - \sin^2(\frac{\alpha\pi}{2})} = 0\end{aligned}$$

and as $k \to 1$, by Lemma 3, $a := k'/4$,

$$\begin{aligned}\log\left(\frac{\Theta(\lambda K)}{\Theta(\mu K)}\right) &\sim \log(1-2a^{2\mu}) - \log(1-2a^{2\lambda})\\ &\quad - 2\log(a)(1-2a^{2\alpha})4a^{2\alpha}\int_0^\beta \frac{\mathrm{d}v}{4(a^{2\alpha}-a^{2v})} + 2\beta(1-\alpha-2a^{2\alpha})\log(a)\\ &\sim 2\log(a)\int_0^\beta \frac{\mathrm{d}v}{a^{2(v-\alpha)}-1} + 2\beta(1-\alpha)\log(a)\\ &= -2\beta\log(a) + \log(a^{2(\beta-\alpha)}-1) - \log(a^{-2\alpha}-1) + 2\beta(1-\alpha)\log(a)\\ &\sim -2\beta\log(a) + 2(\beta-\alpha)\log(a) + 2\alpha\log(a) + 2\beta(1-\alpha)\log(a)\\ &= 2\beta(1-\alpha)\log(a).\end{aligned}$$

If $\lambda > \mu$, note that by the above proved result

$$\begin{aligned}\log\frac{\Theta(\lambda K)}{\Theta(\mu K)} &= -\log\frac{\Theta(\mu K)}{\Theta(\lambda K)} \sim -(\lambda-\mu)(1-(\lambda+\mu)/2)\log a\\ &= (\mu-\lambda)(1-(\lambda+\mu)/2)\log a.\end{aligned}$$

(iii) Obviously, $f(0) = 1$ and, by (1051.02) and (1051.03) of [2],

$$f(1) = \frac{\Theta(\mu K - K)}{\Theta(\mu K + K)} = \frac{\Theta_1(-\mu K)}{\Theta_1(\mu K)} = 1.$$

Further, by Lemma 4,

$$\begin{aligned} f''(\lambda) = K^2 f(\lambda) \Big(& (\mathrm{zn}((\mu - \lambda)K) + \mathrm{zn}((\mu + \lambda)K))^2 \\ & + \mathrm{dn}^2((\mu - \lambda)K) - \mathrm{dn}^2((\mu + \lambda)K) + \frac{2E}{K^2} \Big) > 0 \end{aligned}$$

for every $\lambda \in (0, 1)$, which gives the assertion. □

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